

# Plane-symmetric inhomogeneous Brans-Dicke cosmology with an equation of state $p = \gamma\rho$

Stoytcho S. Yazadjiev \*

Department of Theoretical Physics, Faculty of Physics, Sofia University,  
5 James Bourchier Boulevard, Sofia 1164, Bulgaria

## Abstract

We present a new exact solution in Brans-Dicke theory. The solution describes inhomogeneous plane-symmetric perfect fluid cosmological model with an equation of state  $p = \gamma\rho$ . Some main properties of the solution are discussed.

## 1 Introduction

String theory and higher dimensional unifying theories, in their low energy limit, predict the existence of a scalar partner of the tensor graviton - the dilaton. So obtained scalar-tensor theories are considered as the most natural generalization of general relativity [1]-[3]. A lot of effort has been devoted to the finding and analysis of exact scalar-tensor solutions in order to understand more deeply the physics behind these theories, in particular their relevance to cosmology and astrophysics [2]-[15].

The scalar-tensor gravity equations are much more complicated than the Einstein equations and their solving in the presence of a source is a very difficult task. That is why one should assume some simplifications in order to solve the scalar tensor equations. In this way many homogeneous cosmological solutions with a perfect fluid have been obtained. Some inhomogeneous scalar-tensor cosmologies have also been found and a method for generating general classes of exact scalar-tensor solutions with a stiff perfect fluid has been given [16], [17].

However, the known exact scalar-tensor solutions cover only a small part of the physical content of the scalar-tensor equations. The search of new exact solutions is therefore necessary if further progress is to be made in understanding the scalar-tensor theories. The purpose of this paper is to present a new exact plane-symmetric solution in Brans-Dicke theory with a perfect fluid satisfying the equation of state  $p = \gamma\rho$ . The found solution can be interpreted as an inhomogeneous cosmology. The study of inhomogeneous cosmologies is necessary because, as well known, the present universe is not exactly spatially homogeneous. Although the homogeneous models are

---

\*E-mail: yazad@phys.uni-sofia.bg

good approximations of the present universe there is no reasons to assume that such a regular expansion is also suitable for the description of the early universe. Moreover, it has been shown that the existence of large inhomogeneities in the universe does not necessarily lead to an observable trace left over the spectrum of CMB [18]-[21]. It has also been demonstrated the existence of homogeneous but highly anisotropic cosmological models whose CMB is exactly isotropic [22],[23].

In the light of these results the study of inhomogeneous and anisotropic cosmological models is even imperative.

## 2 Exact solution with a plane symmetry

The general form of the extended gravitational action in scalar-tensor theories is

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-\tilde{g}} \left( F(\Phi) \tilde{R} - Z(\Phi) \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right. \\ \left. - 2U(\Phi) \right) + S_m [\Psi_m; \tilde{g}_{\mu\nu}] . \quad (1)$$

Here,  $G_*$  is the bare gravitational constant,  $\tilde{R}$  is the Ricci scalar curvature with respect to the space-time metric  $\tilde{g}_{\mu\nu}$ . The dynamics of the scalar field  $\Phi$  depends on the functions  $F(\Phi)$ ,  $Z(\Phi)$  and  $U(\Phi)$ . In order for the gravitons to carry positive energy the function  $F(\Phi)$  must be positive. The nonnegativity of the energy of the dilaton field requires that  $2F(\Phi)Z(\Phi) + 3[dF(\Phi)/d\Phi]^2 \geq 0$ . The action of matter depends on the material fields  $\Psi_m$  and the space-time metric  $\tilde{g}_{\mu\nu}$ . It should be noted that the stringy generated scalar-tensor theories, in general, admit the direct interaction between the matter fields and the dilaton in the Jordan (string) frame [1], [2]. Here we consider the phenomenological case when the matter action does not involve the dilaton field in order for the weak equivalence principle to be satisfied.

However, it is much more convenient from a mathematical point of view to analyze the scalar-tensor theories with respect to the conformally related Einstein frame given by the metric:

$$g_{\mu\nu} = F(\Phi) \tilde{g}_{\mu\nu}. \quad (2)$$

Further, let us introduce the scalar field  $\varphi$  (the so called dilaton) via the equation

$$\left( \frac{d\varphi}{d\Phi} \right)^2 = \frac{3}{4} \left( \frac{d \ln(F(\Phi))}{d\Phi} \right)^2 + \frac{Z(\Phi)}{2F(\Phi)} \quad (3)$$

and define

$$\mathcal{A}(\varphi) = F^{-1/2}(\Phi), \quad 2V(\varphi) = U(\Phi)F^{-2}(\Phi). \quad (4)$$

In the Einstein frame action (1) takes the form

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} (R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 4V(\varphi)) + S_m[\Psi_m; \mathcal{A}^2(\varphi)g_{\mu\nu}] \quad (5)$$

where  $R$  is the Ricci scalar curvature with respect to the Einstein metric  $g_{\mu\nu}$ . The Einstein frame field equations then are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 8\pi G_* T_{\mu\nu} + 2\partial_\mu\varphi\partial_\nu\varphi \\ &\quad - g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - 2V(\varphi)g_{\mu\nu} , \\ \nabla^\mu\nabla_\mu\varphi &= -4\pi G_*\alpha(\varphi)T + \frac{dV(\varphi)}{d\varphi} , \\ \nabla_\mu T_\nu^\mu &= \alpha(\varphi)T\partial_\nu\varphi . \end{aligned} \quad (6)$$

Here  $\alpha(\varphi) = d\ln(\mathcal{A}(\varphi))/d\varphi$  and the Einstein frame energy-momentum tensor  $T_{\mu\nu}$  is related to the Jordan frame one  $\tilde{T}_{\mu\nu}$  via  $T_{\mu\nu} = \mathcal{A}^2(\varphi)\tilde{T}_{\mu\nu}$ . In the case of a perfect fluid one has

$$\begin{aligned} \rho &= \mathcal{A}^4(\varphi)\tilde{\rho}, \\ p &= \mathcal{A}^4(\varphi)\tilde{p}, \\ u_\mu &= \mathcal{A}^{-1}(\varphi)\tilde{u}_\mu. \end{aligned} \quad (7)$$

The mathematical complexity makes it very difficult to find inhomogeneous solutions of the system (6) even for simplified models. In fact, when  $g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi < 0$  the system (6) describes two interacting fluids in general relativity. As far as we are aware, there is no scalar-tensor inhomogeneous cosmological solution with an equation of state  $p = \gamma\rho$ . In the particular case  $\gamma = 1/3$  ( $T = 0$ ), we have noninteracting fluids and some inhomogeneous solutions can be found [24].

In this paper we consider Brans-Dicke theory with  $V(\varphi) = 0$ . This theory is described by the functions  $F(\Phi) = \Phi$  and  $Z(\Phi) = \omega/\Phi$  corresponding to  $\mathcal{A}(\varphi) = \exp(\alpha\varphi)$  where  $\alpha = 1/\sqrt{3+2\omega}$ .

We have succeeded in finding the following solution of the field equations (6) for a perfect fluid with an equation of state  $p = \gamma\rho$  ( $\tilde{p} = \gamma\tilde{\rho}$ ) and  $0 < \gamma < 1$ :

$$\begin{aligned} ds^2 &= \cosh^{\frac{1+3\gamma}{\lambda(1-\gamma)}}(\lambda ax)[-d\tau^2 + \mu^2 a^2 \tau^2 dx^2] \\ &\quad + (\mu a\tau)^{\frac{1}{\mu}} \cosh^{-\frac{1}{\lambda}}(\lambda ax)[dy^2 + dz^2], \end{aligned} \quad (8)$$

$$8\pi G_* \rho = \frac{1+\lambda}{\mu^2(1-\gamma)} \frac{1}{\tau^2 \cosh^{2+\frac{1+3\gamma}{\lambda(1-\gamma)}}(\lambda ax)}, \quad (9)$$

$$u_\mu = -\cosh^{\frac{1+3\gamma}{2\lambda(1-\gamma)}}(\lambda ax) \delta_\mu^0, \quad (10)$$

$$\varphi = \frac{1}{2}\alpha \frac{3\gamma-1}{1-\gamma} \left( \frac{1}{\mu} \ln(\mu a\tau) - \frac{1}{\lambda} \ln(\cosh(\lambda ax)) \right), \quad (11)$$

where  $\mu = \mu(\gamma)$  and  $\lambda = \lambda(\gamma)$  are given by

$$2\mu(\gamma) = \frac{2(1-\gamma^2) + \alpha^2(3\gamma-1)^2}{(1-\gamma)^2}, \quad (12)$$

$$\lambda(\gamma) = \frac{(1+3\gamma)(1-\gamma) + \alpha^2(3\gamma-1)^2}{4\gamma(1-\gamma)}. \quad (13)$$

The solution depends on one parameter  $a$  which satisfies  $a > 0$ . The range of the coordinates is

$$0 < \tau < \infty, \quad -\infty < x, y, z < \infty. \quad (14)$$

The spacetime described by the solution is plane-symmetric [25]. It has a three dimensional group of local isometries<sup>1</sup> acting on two-dimensional orbits and generated by the Killing vectors:

$$\mathcal{K}_1 = \frac{\partial}{\partial y}, \quad \mathcal{K}_2 = \frac{\partial}{\partial z}, \quad \mathcal{K}_3 = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}. \quad (15)$$

Therefore, the found solution can be interpreted as an inhomogeneous cosmology.

Let us note that in the particular case  $\gamma = 1/3$  we obtain pure general relativistic solution with a trivial dilaton field  $\varphi = 0$ .

As  $\tau \rightarrow 0$  the curvature invariants, energy density, pressure and the dilaton field diverge which corresponds to a big-bang singularity. The Einstein frame perfect fluid expansion, acceleration and shear calculated in the natural orthonormal tetrad

$$e^0 = |g_{00}|^{1/2} dt, \quad e^1 = g_{11}^{1/2} dx, \quad e^2 = g_{22}^{1/2} dy, \quad e^3 = g_{33}^{1/2} dz, \quad (16)$$

are the following:

---

<sup>1</sup>Obviously, there is also a discrete isometry given by  $x \rightarrow -x$ .

$$\theta = (\mu + 1) \frac{a}{g_{11}^{1/2}}, \quad (17)$$

$$a_1 = \frac{1+3\gamma}{2(1-\gamma)} a \frac{\tanh(\lambda a x)}{g_{11}^{1/2}}, \quad (18)$$

$$\sigma_{11} = \frac{2\mu-1}{3} \frac{a}{g_{11}^{1/2}}, \quad (19)$$

$$\sigma_{22} = \sigma_{33} = -\frac{1}{2} \sigma_{11}. \quad (20)$$

It is seen that the fluid is everywhere expanding in the Einstein frame. It is also interesting to note that, near the singularity, the scalar fluid is not dominant ( $\rho_\varphi \sim 1/\tau^2$ ) due to the interaction between the fluids.

The Jordan (string) frame solution can be obtained by using the inverse of transformations (2), (3),(4) and (7).

The properties of the Jordan frame solution depend on the value of the parameter  $\alpha$ . In the physically interesting case when  $\alpha \leq 1$  (i.e. when  $\omega$  is equal to or greater than the stringy value  $\omega = -1$ ) the Jordan frame solution has qualitatively the same properties as the Einstein frame solution.

With regard to the kinematical quantities in the Jordan frame, the only non-vanishing components of expansion, acceleration and shear of the fluid are given by:

$$\tilde{\theta} = \frac{2(1-\gamma) + \alpha^2(3\gamma-1)}{(1-\gamma)^2} \frac{a}{\tilde{g}_{11}^{1/2}}, \quad (21)$$

$$\tilde{a}_1 = \frac{(1+3\gamma) + \alpha^2(1-3\gamma)}{2(1-\gamma)} a \frac{\tanh(\lambda a x)}{\tilde{g}_{11}^{1/2}}, \quad (22)$$

$$\tilde{\sigma}_{11} = \frac{2\mu-1}{3} \frac{a}{\tilde{g}_{11}^{1/2}}, \quad (23)$$

$$\tilde{\sigma}_{22} = \tilde{\sigma}_{33} = -\frac{1}{2} \tilde{\sigma}_{11}. \quad (24)$$

All quantities are calculated in the natural orthonormal tetrad:

$$\tilde{e}^0 = |\tilde{g}_{00}|^{1/2} dt, \quad \tilde{e}^1 = \tilde{g}_{11}^{1/2} dx, \quad \tilde{e}^2 = \tilde{g}_{22}^{1/2} dy, \quad \tilde{e}^3 = \tilde{g}_{33}^{1/2} dz. \quad (25)$$

To the best of our knowledge the presented exact solution is the first example of an inhomogeneous perfect fluid scalar-tensor cosmology with an equation of state different from that of the stiff matter.

## Acknowledgments

This work was partially supported by Sofia University Grant No.3429/2002-03.

## References

- [1] T.Damour, A. Polyakov, Nucl. Phys. **B 423**, 532 (1994)
- [2] M. Gasperini, G. Veneziano, The Pre-Big Bang Scenario in String Cosmology, hep-th/0207130
- [3] M. Gasperini's web page, <http://www.to.infn.it/~gasperin>
- [4] J. Barrow, K. Maeda, Nucl. Phys. **B 341**, 294 (1990)
- [5] J. Barrow, Phys. Rev. **D 47**, 5329 (1993)
- [6] J. Barrow, J. Mimoso, Phys. Rev. **D 50**, 3746 (1994)
- [7] J. Mimoso, D. Wands, Phys. Rev. **D 51**, 477 (1995)
- [8] J. Mimoso, D. Wands, Phys. Rev. **D 52**, 5612 (1995)
- [9] P. Chauvet, J. Cervantes-Cota, Phys. Rev. **D 52**, 3416 (1995)
- [10] L. Pimentel, Phys. Rev. **D 53**, 1808 (1995)
- [11] M. Giovannini, Phys. Rev. **D 57**, 7223 (1998)
- [12] M. Giovannini, Phys. Rev. **D 59**, 083511 (1999)
- [13] L. Pimentel, Mod. Phys. Lett **A**, Vol **14**, 43 (1999)
- [14] J. Cervantes-Cota, M. Nahmad, Gen.Rel.Grav. **33**, 767 (2001)
- [15] S. Yazadjiev, A class of homogeneous scalar-tensor cosmologies with a radiation fluid, gr-qc/0211093
- [16] S. Yazadjiev, Phys. Rev. **D 65**, 084023 (2002)
- [17] S. Yazadjiev, Phys. Rev. **D 66**, 024031 (2002)
- [18] D. Raine, E. Thomas, Mon. Not. Roy. Astr. Soc. **195**, 649 (1981)
- [19] M. Panek, Astrophys. J. **388**, 225 (1992)
- [20] D. Saez, J. Arnau, M. Fullana, Mon. Not. Roy. Astr. Soc. **263**, 681 (1993)
- [21] J. Arnau, M. Fullana, D. Saez, Mon. Not. Roy. Astr. Soc. **268**, L17 (1994)
- [22] W. Lim, U. Nilsson, J. Wainwright, Class. Quant. Grav. **18**, 5583 (2001)
- [23] S. Carneiro, G. Mena Marugan, Phys. Rev. **D 64**, 083502 (2001)
- [24] S. Yazadjiev, unpublished
- [25] D. Kramer, H. Stephani, E. Herlt, M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 1980)